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Singular boundary value problems of fractional differential equations with changing sign nonlinearity and parameter

Yong Mu¹, Liying Sun² and Zhenlai Han^{2*}

*Correspondence:

hanzhenlai@163.com

²School of Mathematical Sciences,
University of Jinan, Jinan, Shandong
250022, P.R. ChinaFull list of author information is
available at the end of the article

Abstract

In this paper, we consider singular boundary value problems for the following nonlinear fractional differential equations with delay:

$$\begin{cases} D^\alpha x(t) + \lambda f(t, x(t-\tau)) = 0, & t \in (0, 1) \setminus \{\tau\}, \\ x(t) = \eta(t), & t \in [-\tau, 0], \\ x'(1) = x'(0) = 0, \end{cases}$$

where $2 < \alpha \leq 3$, D^α denotes the Riemann-Liouville fractional derivative, λ is a positive constant, $f(t, x)$ may change sign and be singular at $t = 0$, $t = 1$, and $x = 0$. By means of the Guo-Krasnoselskii fixed point theorem, the eigenvalue intervals of the nonlinear fractional functional differential equation boundary value problem are considered, and some positive solutions are obtained, respectively.

MSC: 34A08; 34K37**Keywords:** fractional functional differential equation; delay; boundary value problems; singular; positive solutions

1 Introduction

Fractional differential equations have been of increasing importance for the past decades due to their diverse applications in science and engineering, we can describe natural phenomena and mathematical models more accurately. Many researchers have shown their interest in fractional differential equations. The motivation for those works stems from both the intensive development of the theory of fractional calculus itself and the applications such as in economics, engineering and other fields. Fractional differential equations have received much attention, the theory and its applications have been greatly developed; see [1–5].

There have been many papers dealing with boundary value problems of fractional differential equations [6–22] and initial value problems of fractional differential equations [23–31].

However, the results focused on the singular boundary value problems of fractional differential equations with delay are relatively scarce [32–37]. It is well known that in practical

problems, the behavior of systems not only depends on the status just at the present, but also on the status in the past.

Thus, in many cases, we must study fractional differential equations with delay in order to solve practical problems. Consequently, our aim in the paper is to consider the existence of solutions for singular boundary value problems of fractional differential equations with delay.

In 2004, by means of the fixed point index theorem, Yu *et al.* [38] investigate the existence of multiple positive solutions for the third-order three-point singular semipositone boundary value problem

$$\begin{cases} x'''(t) - \lambda f(t, x) = 0, & t \in (0, 1), \\ x(0) = x'(\eta) = x''(1) = 0, \end{cases}$$

where $\frac{1}{2} < \eta < 1$, $f(t, x) : (0, 1) \times (0, +\infty) \rightarrow (-\infty, +\infty)$ is continuous and may be singular at $t = 0$, $t = 1$, and $x = 0$ and also may be negative for some values of t and x ; λ is a positive parameter.

In 2011, Zhao *et al.* [12] studied the existence on multiple positive solutions for the nonlinear fractional differential equation boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

where $2 < \alpha \leq 3$, D_{0+}^{α} is the Riemann-Liouville fractional derivative. By the lower and upper solutions method and the fixed point theorem, they obtained some new existence criteria for singular and nonsingular fractional differential equation boundary value problems.

In 2012, Su [37] studied the boundary value problem for a singular fractional differential equation with delay

$$\begin{cases} D^{\alpha} x(t) + f(t, x(t - \tau)) = 0, & t \in (0, 1) \setminus \{\tau\}, \\ x(t) = \eta(t), & t \in [-\tau, 0], \\ x(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2$, D^{α} is the Riemann-Liouville fractional derivative, $\tau \in (0, 1)$, $f(t, x) \in C((0, 1) \times \mathbb{R}^+, \mathbb{R})$ is continuous and may be singular at $t = 0$, $t = 1$, and $x = 0$ and may have negative values, where $\mathbb{R}^+ = (0, +\infty)$. By the Guo-Krasnoselskii fixed point theorem, one obtained the existence results for positive solutions.

In 2013, Vong [39] considered the fractional differential equation with an integral boundary condition

$${}^c D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad u'(0) = \dots = u^{(n-1)}(0) = 0, \quad u(1) = \int_0^1 u(s) d\mu(s),$$

where $n \geq 2$, $n - 1 < \alpha < n$, ${}^c D_{0+}^{\alpha}$ is the Caputo fractional derivative, $\mu(s)$ is a function of bounded variation, f may have a singularity at $t = 1$. The existence of positive solutions is obtained by the method of upper and lower solutions together with the Schauder fixed point theorem.

Motivated by the work mentioned above, in this paper, we study the existence of positive solutions of singular boundary value problems for nonlinear fractional functional differential equation

$$\begin{cases} D^\alpha x(t) + \lambda f(t, x(t-\tau)) = 0, & t \in (0, 1) \setminus \{\tau\}, \\ x(t) = \eta(t), & t \in [-\tau, 0], \\ x'(1) = x'(0) = 0, \end{cases} \quad (1.1)$$

where $2 < \alpha \leq 3$, D^α denote the Riemann-Liouville fractional derivative, λ is a positive parameter, $\tau \in (0, 1)$, $\eta(t) \in C([-\tau, 0])$, and $\eta(t) > 0$ for $t \in [\tau, 0)$, $\eta(0) = 0$, f is a continuous functional defined on $(0, 1) \times \mathbb{R}^+$ and which may be singular at $t = 0$, $t = 1$, and $x = 0$.

When $\tau = 0$ and $\lambda = 1$, problem (1.1) is reduced to the problem of fractional differential equations and has been studied by Zhao *et al.* [12]. To the best of our knowledge, no one has studied the existence of positive solutions for singular boundary value problem (1.1). Key tools in finding our main results are the Guo-Krasnoselskii fixed point theorem, and our main results of this paper are to extend and supplement some results in [12, 37, 38].

The paper is organized as follows. In Section 2, we shall introduce some definitions and lemmas to prove our main results. In Section 3, we investigate the existence of positive solution for boundary value problem (1.1) by the Guo-Krasnoselskii fixed point theorem.

2 Preliminaries

In the following section, we introduce the definitions and lemmas which are used throughout the paper. This material can be found in [1, 2].

The Riemann-Liouville fractional derivative of order α ($n-1 < \alpha < n$) of a function $f : (t_0, +\infty) \rightarrow \mathbb{R}$ is given by

$$D_{0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_{t_0}^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \quad t > t_0,$$

where n is the smallest integer than or equal to α and $\Gamma(\cdot)$ is the gamma function, provided that the right side is point wise defined on $(t_0, +\infty)$.

The Riemann-Liouville fractional integral of order α ($\alpha > 0$) of a function $f : (t_0, +\infty) \rightarrow \mathbb{R}$ is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > t_0,$$

where $\Gamma(\cdot)$ is the gamma function, provided that the right side is point-wise defined on $(t_0, +\infty)$.

From the definition of the Riemann-Liouville derivative, we have the following statements.

Let $\alpha > 0$. If we assume $u \in C(0, 1) \cap L(0, 1)$, then the fractional differential equation

$$D_{0+}^\alpha u(t) = 0$$

has $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_N t^{\alpha-N}$, $c_i \in \mathbb{R}$, $i = 1, 2, \dots, N$, as unique solutions, where N is the smallest integer greater than or equal to α .

Assume $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$. Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = f(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, where n is the smallest integer greater than or equal to α .

Next we introduce the Green function of boundary value problems for fractional differential equations.

Lemma 2.1 [12] *Let $2 < \alpha \leq 3$ and $h : [0, 1]$ be continuous. Then the unique solution of the boundary value problem*

$$D_{0+}^{\alpha} u(t) + \lambda h(t) = 0, \quad t \in (0, 1), \quad u(0) = u'(0) = u'(1) = 0 \quad (2.1)$$

is

$$u(t) = \int_0^1 \lambda G(t, s) h(s) ds, \quad t \in [0, 1],$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (2.2)$$

The following properties of the Green function play important roles in this paper.

Lemma 2.2 [12] *The function $G(t, s)$ defined by (2.2) satisfies the following conditions:*

- (1) $G(t, s) > 0$ for $t, s \in (0, 1)$;
- (2) $q(t)G(1, s) \leq G(t, s) \leq G(1, s)$ for $t, s \in (0, 1)$, where $q(t) = t^{\alpha-1}$.

Lemma 2.3 *The function $G^*(t, s) := t^{5-\alpha} G(t, s)$ satisfies the following conditions:*

$$\frac{1}{\Gamma(\alpha)} t^4 s(1-s)^{\alpha-2} \leq G^*(t, s) \leq \frac{1}{\Gamma(\alpha)} t^{5-\alpha} s(1-s)^{\alpha-2} \quad \text{for } t, s \in (0, 1).$$

Proof The proof can be obtained easily by Lemma 2.2, so we omit it here. \square

The following lemma is fundamental in the proofs of our main results.

Lemma 2.4 [40] *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open and bounded subset of E with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$ be a completely continuous operator such that*

- (i) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$; or
- (ii) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Main results

In this section, we discuss the existence of positive solutions for boundary value problem (1.1). For convenience, we give some conditions, which will play roles in this paper.

(H₁) There exists a nonnegative function $\rho \in C(0,1) \cap L(0,1)$ such that

$$\varphi_2(t)h_2(x) \leq f(t, v(t)x) + \rho(t) \leq \varphi_1(t)(g(x) + h_1(x))$$

for all $(t, x) \in (0,1) \times \mathbb{R}^+$, where $\varphi_1, \varphi_2 \in L(0,1)$ are nonnegative for $t \in (0,1)$, $h_1, h_2 \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ are nondecreasing, $g \in C(\mathbb{R}^+, \mathbb{R}_0^+)$ is nonincreasing, $\mathbb{R}_0^+ = [0, +\infty)$, and

$$v(t) = \begin{cases} 1, & t \in (0, \tau], \\ (t - \tau)^{\alpha-5}, & t \in (\tau, 1). \end{cases}$$

(H₂)

$$0 < \int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) g(\eta(s-\tau)) ds < +\infty,$$

and there exists a constant $k > 0$ such that

$$\int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) g\left(\frac{k}{2}(s-\tau)^2\right) ds < +\infty.$$

(H₃) There exists a subinterval $[a, b] \subset (\tau, 1)$ such that $\int_a^b s(1-s)^{\alpha-1} \varphi_2(s) ds > 0$.

Let $X := \{x(t) : x \in C([- \tau, 1], \mathbb{R}), x(t) = 0 \text{ for } t \in [- \tau, 0], x'(1) = x'(0) = 0\}$ be a Banach space with the maximum norm $\|x\|_{[- \tau, 1]} = \max_{- \tau \leq t \leq 1} |x(t)| = \max_{0 \leq t \leq 1} |x(t)|$ for $x \in X$. Let K be a cone in X defined by

$$K = \{x \in X; x(t) \geq 0 \text{ for } t \in [- \tau, 1]\}.$$

Define

$$\bar{\eta}(t) = \begin{cases} \eta(t), & t \in [- \tau, 0], \\ 0, & t \in (0, 1], \end{cases}$$

$$\omega(t) = \begin{cases} 0, & t \in [- \tau, 0], \\ \int_0^1 \lambda G(t, s) \rho(s) ds, & t \in (0, 1], \end{cases}$$

and

$$x^*(t) = \max\{x(t) + \bar{\eta}(t) - \omega(t), 0\}$$

$$= \begin{cases} \eta(t), & t \in [- \tau, 0], \\ \max\{x(t) - \omega(t), 0\}, & t \in (0, 1] \end{cases}$$

for any $x \in K$.

It is easy to know that the restriction $\omega|_{[0,1]}$ of ω on $[0,1]$ is exactly the solution of

$$\begin{cases} D^\alpha x(t) + \lambda \rho(t) = 0, & t \in (0, 1), \alpha \in (2, 3], \\ x'(1) = x'(0) = x(0) = 0. \end{cases}$$

Since $f : [0, 1] \times C[-\tau, 1] \rightarrow \mathbb{R}$ is a continuous function, setting $f(t, x(t - \tau)) := h(t)$ in Lemma 2.1, we see by Lemma 2.1 that a function x is a solution of boundary value problem (1.1) if and only if it satisfies

$$x(t) = \begin{cases} \int_0^1 \lambda G(t, s) f(s, x(s - \tau)) ds, & t \in (0, 1), \\ \eta(t), & t \in [-\tau, 0]. \end{cases}$$

Consider the following operator:

$$(Tx)(t) = \begin{cases} \int_0^1 \lambda G(t, s) (f(s, x^*(s - \tau)) + \rho(s)) ds, & t \in (0, 1), \\ 0, & t \in [-\tau, 0]. \end{cases} \quad (3.1)$$

Set

$$y(t) = \begin{cases} t^{5-\alpha} x(t), & t \in (0, 1), \\ 0, & t \in [-\tau, 0] \end{cases}$$

and

$$y^*(t) = \begin{cases} \max\{t^{\alpha-5} y(t) - \omega(t), 0\}, & t \in (0, 1], \\ \eta(t), & t \in [-\tau, 0]. \end{cases}$$

Then (3.1) is equivalent to

$$(Ty)(t) = \begin{cases} \int_0^1 \lambda G^*(t, s) (f(s, y^*(s - \tau)) + \rho(s)) ds, & t \in (0, 1], \\ 0, & t \in [-\tau, 0]. \end{cases} \quad (3.2)$$

Obviously, if \tilde{y} is a fixed point of operator T in (3.2), then

$$\tilde{x}(t) = \begin{cases} t^{\alpha-5} \tilde{y}(t), & t \in (0, 1], \\ 0, & t \in [-\tau, 0], \end{cases}$$

is a fixed point of operator T defined by (3.1). Lemma 2.1 implies that

$$\begin{cases} D^\alpha \tilde{x}(t) + \lambda f((t, \tilde{x}^*(t - \tau)) + \rho(t)) = 0, & t \in (0, 1) \setminus \{\tau\}, \\ \tilde{x}(t) = 0, & t \in [-\tau, 0], \\ \tilde{x}'(1) = \tilde{x}'(0) = 0. \end{cases}$$

Thus if

$$\tilde{x}(t - \tau) + \bar{\eta}(t - \tau) - \omega(t - \tau) \geq 0 \quad \text{for } t \in [0, 1], \quad (3.3)$$

then

$$\tilde{x}^*(t - \tau) = \tilde{x}(t - \tau) + \bar{\eta}(t - \tau) - \omega(t - \tau).$$

Let

$$x(t) = \tilde{x}(t) + \bar{\eta}(t) - \omega(t).$$

Then one can verify easily that the function x satisfied boundary value problem (1.1). As a result, in the following we will concentrate our study on finding the fixed points of operator T defined by (3.2).

Define the cone

$$K_1 = \{y \in K : y(t) \geq t^2 \|y\| \text{ for } t \in [0, 1]\}$$

and

$$\Omega_1 = \{y \in K_1 : \|y\| < r_1\},$$

$$\Omega_2 = \{y \in K_1 : \|y\| < r_2\},$$

$$\Omega_3 = \{y \in K_1 : \|y\| < R_2\},$$

$$\Omega_4 = \{y \in K_1 : \|y\| < R_1\}$$

for any $r_2 > r_1 \geq \max\{k, 2c\}$, $R_2 > R_1 \geq \max\{k, 2c\}$, where

$$c := \frac{\lambda}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} \rho(s) ds < +\infty \quad (3.4)$$

and k is the constant in (H_2) .

Lemma 3.1 *Let (H_1) and (H_2) hold. Then the operator $T : K_1 \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K_1$ is completely continuous.*

Proof First we show that operator T is well defined on $K_1 \cap (\bar{\Omega}_2 \setminus \Omega_1)$. For any $y \in K_1 \cap (\bar{\Omega}_2 \setminus \Omega_1)$, we know that

$$r_1 \leq \|y\| \leq r_2$$

and

$$y(t) \geq t^2 \|y\| \geq t^2 r_1 \quad \text{for } t \in [0, 1].$$

Then, for $t \in [0, 1]$, we get

$$\begin{aligned} t^{5-\alpha} \omega(t) &= t^{5-\alpha} \int_0^1 \lambda G(t, s) \rho(s) ds \\ &\leq \frac{t^{5-\alpha}}{\Gamma(\alpha)} \int_0^1 \lambda s(1-s)^{\alpha-2} \rho(s) ds \\ &\leq \frac{t^2}{\Gamma(\alpha)} \int_0^1 \lambda s(1-s)^{\alpha-2} \rho(s) ds \\ &\leq t^2 c, \end{aligned} \quad (3.5)$$

where c is defined as (3.4). Thus, for $t \in [0, 1]$,

$$\begin{aligned} y(t) - t^{5-\alpha} \omega(t) &\geq t^2 (r_1 - c) \\ &\geq \frac{r_1}{2} t^2. \end{aligned} \quad (3.6)$$

In view of (H_1) , (H_2) , and Lemma 2.3, we show

$$\begin{aligned}
 (Ty)(t) &= \int_0^\tau \lambda G^*(t,s) (f(s, \eta(s-\tau)) + \rho(s)) ds \\
 &\quad + \int_\tau^1 \lambda G^*(t,s) (f(s, (s-\tau)^{\alpha-5} y(s-\tau) - \omega(s-\tau)) + \rho(s)) ds \\
 &\leq \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \\
 &\quad + \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) \\
 &\quad \times \left(g\left(\frac{r_1}{2}(s-\tau)^2\right) + h_1(y(s-\tau) - (s-\tau)^{5-\alpha} \omega(s-\tau)) \right) ds \\
 &\leq \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \\
 &\quad + \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{r_1}{2}(s-\tau)^2\right) + h_1(y(s-\tau)) \right) ds \\
 &\leq \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \\
 &\quad + \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{k}{2}(s-\tau)^2\right) + h_1(r_2) \right) ds \\
 &< +\infty.
 \end{aligned}$$

Hence, T is uniformly bounded and T is well defined.

In fact, for $y \in K_1 \cap (\bar{\Omega}_2 \setminus \Omega_1)$, $t \in [0, 1]$, in view of Lemma 2.3, we have

$$\begin{aligned}
 \|Ty\| &\leq \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} (f(s, y^*(s-\tau)) + \rho(s)) ds \\
 &\leq \frac{\lambda t^2}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} (f(s, y^*(s-\tau)) + \rho(s)) ds
 \end{aligned}$$

and

$$\begin{aligned}
 (Ty)(t) &\geq \frac{\lambda t^4}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} (f(s, y^*(s-\tau)) + \rho(s)) ds \\
 &\geq t^2 \|Ty\|.
 \end{aligned}$$

Hence, $T : K_1 \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K_1$.

Next we show $T : K_1 \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K_1$ is continuous and compact. For any $y_n, y \in K_1 \cap (\bar{\Omega}_2 \setminus \Omega_1)$, $n = 1, 2, \dots$ with $\|y_n - y\|_{[-\tau, 1]} \rightarrow 0$ as $n \rightarrow \infty$. Since $r_1 \leq \|y_n\| \leq r_2$ and $r_1 \leq \|y\| \leq r_2$, for $t \in (0, 1)$, we know

$$y_n(t) - t^{5-\alpha} \omega(t) \geq \frac{r_1}{2} t^2$$

and

$$y(t) - t^{5-\alpha} \omega(t) \geq \frac{r_1}{2} t^2.$$

Then, for $t \in [0, 1]$, we have

$$\begin{aligned} |(Ty_n)(t) - (Ty)(t)| &= \left| \int_{\tau}^1 \lambda G^*(t, s) (f(s, (s - \tau)^{\alpha-5} y_n(s - \tau) - \omega(s - \tau)) + \rho(s) \right. \\ &\quad \left. - f(s, (s - \tau)^{\alpha-5} y(s - \tau) - \omega(s - \tau)) - \rho(s)) ds \right| \\ &\leq \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_{\tau}^1 s(1-s)^{\alpha-2} |f(s, (s - \tau)^{\alpha-5} y_n(s - \tau) - \omega(s - \tau)) \\ &\quad - f(s, (s - \tau)^{\alpha-5} y(s - \tau) - \omega(s - \tau))| ds \\ &\leq \frac{2\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_{\tau}^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{k}{2}(s - \tau)^2\right) + h_1(r_2) \right) ds \\ &< +\infty. \end{aligned}$$

This implies that $\|Ty_n - Ty\|_{[-\tau, 1]} \rightarrow 0$ as $n \rightarrow \infty$. Hence T is continuous.

Next we prove T is equicontinuous.

Since G^* is uniformly continuous for $t \in (0, 1)$, that is, for any $\epsilon > 0$, there exists $\xi_0 > 0$, when $t_1, t_2 \in [0, 1]$ and $|t_1 - t_2| < \xi_0$, we have

$$\begin{aligned} |G^*(t_1, s) - G^*(t_2, s)| &= \frac{\epsilon}{2} \left(\int_0^{\tau} \lambda \varphi_1(s) (g(\eta(s - \tau)) + h_1(\eta(s - \tau))) ds \right. \\ &\quad \left. + \int_{\tau}^1 \lambda \varphi_1(s) \left(g\left(\frac{k}{2}(s - \tau)^2\right) + h_1(r_2) \right) ds \right)^{-1}. \end{aligned}$$

Thus, for any $y \in K_1 \cap (\bar{\Omega}_2 \setminus \Omega_1)$, we get

$$\begin{aligned} |(Ty)(t_1) - (Ty)(t_2)| &\leq \int_0^{\tau} \lambda |G^*(t_1, s) - G^*(t_2, s)| \varphi_1(s) (g(\eta(s - \tau)) + h_1(\eta(s - \tau))) ds \\ &\quad + \int_{\tau}^1 \lambda |G^*(t_1, s) - G^*(t_2, s)| \varphi_1(s) \left(g\left(\frac{k}{2}(s - \tau)^2\right) + h_1(r_2) \right) ds \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus T is equicontinuous. Accordingly to the Ascoli-Arzelà theorem, T is completely continuous. The proof is completed. \square

Now we prove the existence of positive solutions for boundary value problem (1.1) by using the Guo-Krasnoselskii fixed point theorem.

For convenience, we denote

$$\begin{aligned} \xi_1 &:= \frac{1}{\Gamma(\alpha)} \left(\int_0^{\tau} s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s - \tau)) \right. \\ &\quad \left. + h_1(\eta(s - \tau))) ds + \int_{\tau}^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{k}{2}(s - \tau)^2\right) + \epsilon r_2 \right) ds \right) \\ &> 0, \end{aligned}$$

and there exists a subinterval $[\beta, \gamma] \subset (\tau, 1)$,

$$\begin{aligned}\zeta_1 &:= \min_{t \in [\beta, \gamma]} (t - \tau)^2, & \zeta_2 &:= \min_{t \in [\beta, \gamma]} t^4, \\ \xi_2(t) &:= \frac{\zeta_2}{\Gamma(\alpha)} h_2 \left(\frac{r_1 \zeta_1}{2} \right) \int_{\beta}^{\gamma} s(1-s)^{\alpha-2} \varphi_2(s) ds > 0.\end{aligned}$$

Theorem 3.1 *Let (H_1) , (H_2) , and $\xi_2^{-1}r_1 < \xi_1^{-1}r_2$ hold. Then the boundary value problem (1.1) has at least one positive solution if*

$$\lim_{y \rightarrow +\infty} \frac{h_1(y)}{y} = 0 \quad (3.7)$$

for each

$$\lambda \in (\xi_2^{-1}r_1, \xi_1^{-1}r_2).$$

Proof Let $\epsilon > 0$. Then in view of (3.7), there exists a $M > 0$ such that

$$h_1(y) \leq \epsilon y \quad \text{for } y > M. \quad (3.8)$$

Choose

$$r_2 \geq \max\{M + 1, r_1 + 1\},$$

then for $y \in \partial\Omega_2$, like for (3.6), for $t \in [0, 1]$, we obtain

$$\begin{aligned}y(t) - t^{5-\alpha}\omega(t) &\geq t^2(r_2 - c) \\ &\geq \frac{r_2}{2}t^2.\end{aligned} \quad (3.9)$$

Then from (H_1) , (3.8), (3.9), and Lemma 2.3, we get

$$\begin{aligned}(Ty)(t) &\leq \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_0^{\tau} s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \\ &\quad + \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_{\tau}^1 s(1-s)^{\alpha-2} \varphi_1(s) \\ &\quad \times \left(g\left(\frac{r_2}{2}(s-\tau)^2\right) + h_1(y(s-\tau) - (s-\tau)^{5-\alpha}\omega(s-\tau)) \right) ds \\ &\leq \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_0^{\tau} s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \\ &\quad + \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_{\tau}^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{r_2}{2}(s-\tau)^2\right) + h_1(y(s-\tau)) \right) ds \\ &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^{\tau} s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \\ &\quad + \frac{\lambda}{\Gamma(\alpha)} \int_{\tau}^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{k}{2}(s-\tau)^2\right) + h_1(r_2) \right) ds\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{k}{2}(s-\tau)^2\right) + \epsilon r_2 \right) ds \\
&\leq \lambda \xi_1 < r_2.
\end{aligned}$$

Therefore, for $y \in \partial\Omega_2$, we have $\|Ty\| \leq \|y\|$.

On the other hand, for $y \in \partial\Omega_1$, like for (3.6), for $t \in [0, 1]$, we obtain

$$\begin{aligned}
y(t) - t^{5-\alpha} \omega(t) &\geq t^2(r_1 - c) \\
&\geq \frac{r_1}{2} t^2.
\end{aligned} \tag{3.10}$$

Thus from (H_1) , (3.10), and Lemma 2.3, we get

$$\begin{aligned}
\|Ty\| &\geq \int_\beta^\gamma \lambda \min_{t \in [\beta, \gamma]} G^*(t, s) (f(s, (s-\tau)^{\alpha-5} y(s-\tau) - \omega(s-\tau)) + \rho(s)) ds \\
&\geq \int_\beta^\gamma \lambda \min_{t \in [\beta, \gamma]} G^*(t, s) \varphi_2(s) h_2\left(\frac{r_1}{2}(s-\tau)^2\right) ds \\
&\geq \frac{\lambda \xi_2}{\Gamma(\alpha)} h_2\left(\frac{r_1 \xi_1}{2}\right) \int_\beta^\gamma s(1-s)^{\alpha-2} \varphi_2(s) ds \\
&\geq \lambda \xi_2 > r_1.
\end{aligned}$$

Therefore, for $y \in \partial\Omega_1$, we have $\|Ty\| \geq \|y\|$. Then T defined by (3.2) has a fixed point $\tilde{y} \in K_1 \cap (\bar{\Omega}_2 \setminus \Omega_1)$. In view of (3.10), we have

$$\begin{aligned}
t^{\alpha-5} \tilde{y}(t) - \omega(t) &= t^{\alpha-5} (\tilde{y}(t) - t^{5-\alpha} \omega(t)) \\
&\geq \frac{r_1}{2} t^{\alpha-3} \\
&> 0.
\end{aligned}$$

It is easy to know (3.3) is satisfied. The proof is completed. \square

Denote

$$\begin{aligned}
\xi_3 &:= \frac{1}{\Gamma(\alpha)} \left(\int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \right. \\
&\quad \left. + \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{k}{2}(s-\tau)^2\right) + h_1(R_1) \right) ds \right) \\
&> 0.
\end{aligned}$$

Theorem 3.2 *Let (H_1) – (H_3) and*

$$\frac{2\Gamma(\alpha)}{M^* A \xi_2} \left(\int_a^b s(1-s)^{\alpha-2} \varphi_2(s) ds \right)^{-1} < \xi_3^{-1} R_1$$

hold. Then boundary value problem (1.1) has at least one positive solution if

$$\lim_{y \rightarrow +\infty} \frac{h_2(y)}{y} = +\infty \quad (3.11)$$

for each

$$\lambda \in \left(\frac{2\Gamma(\alpha)}{M^*A\xi_2} \left(\int_a^b s(1-s)^{\alpha-2} \varphi_2(s) ds \right)^{-1}, \xi_3^{-1} R_1 \right), \quad (3.12)$$

where $A := \min_{t \in [a,b]} (t - \tau)^2$, M^* is a positive constant.

Proof It follows from (3.11) that there exists a $M^* > 0$ such that

$$h_2(y) \geq M^*y \quad \text{for } y > M^*. \quad (3.13)$$

Choose

$$R_2 \geq \max \left\{ R_1 + 1, \frac{2M^*}{A} \right\}.$$

Then for $y \in \partial\Omega_3$, like for (3.6), for $t \in [0, 1]$, we obtain

$$\begin{aligned} y(t) - t^{5-\alpha} \omega(t) &\geq t^2(R_2 - c) \\ &\geq \frac{R_2}{2} t^2. \end{aligned} \quad (3.14)$$

Thus from (H_1) , (3.13), (3.14), and Lemma 2.3, we get

$$\begin{aligned} \|Ty\| &\geq \int_a^b \lambda \min_{t \in [a,b]} G^*(t, s) (f(s, (s - \tau)^{\alpha-5} y(s - \tau) - \omega(s - \tau)) + \rho(s)) ds \\ &\geq \int_a^b \lambda \min_{t \in [a,b]} G^*(t, s) \varphi_2(s) h_2 \left(\frac{R_2}{2} (s - \tau)^2 \right) ds \\ &\geq \int_a^b \lambda \min_{t \in [a,b]} G^*(t, s) \varphi_2(s) h_2 \left(\frac{R_2 A}{2} \right) ds \\ &\geq \frac{M^* A \lambda R_2}{2} \int_a^b \min_{t \in [a,b]} G^*(t, s) \varphi_2(s) ds \\ &\geq \frac{\lambda M^* A R_2 \xi_2}{2\Gamma(\alpha)} \int_a^b s(1-s)^{\alpha-2} \varphi_2(s) ds \\ &\geq R_2. \end{aligned}$$

Therefore, for $y \in \partial\Omega_3$, we have $\|Ty\| \geq \|y\|$.

On the other hand, for $y \in \partial\Omega_4$, like for (3.6), for $t \in [0, 1]$, we obtain

$$\begin{aligned} y(t) - t^{5-\alpha} \omega(t) &\geq t^2(R_1 - c) \\ &\geq \frac{R_1}{2} t^2. \end{aligned} \quad (3.15)$$

Thus from (H_1) , (3.12), and (3.15), we have

$$\begin{aligned}(Ty)(t) &\leq \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \\ &\quad + \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) \\ &\quad \times \left(g\left(\frac{R_1}{2}(s-\tau)^2\right) + h_1(y(s-\tau) - (s-\tau)^{5-\alpha} \omega(s-\tau)) \right) ds \\ &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \\ &\quad + \frac{\lambda}{\Gamma(\alpha)} \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{k}{2}(s-\tau)^2\right) + h_1(R_1) \right) ds \\ &\leq \lambda \xi_3 < R_1.\end{aligned}$$

Therefore, for $y \in \partial\Omega_4$, we have $\|Ty\| \leq \|y\|$. Arguments similar to those at the end of the proof of Theorem 3.1 show that boundary value problem (1.1) has a positive solution. The proof is completed. \square

Theorem 3.3 *Let (H_1) and (H_2) hold. Furthermore assume that*

(H_4) *there exists a subinterval $[\beta, \gamma] \subset (\tau, 1)$ and a positive constant r such that*

$$\begin{aligned}r &> \max \left\{ k, 2c, \frac{\lambda}{\Gamma(\alpha)} \left(\int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \right. \right. \\ &\quad \left. \left. + \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{k}{2}(s-\tau)^2\right) + h_1(r) \right) ds \right) \right\},\end{aligned}$$

where k is defined in (H_2) , c is defined as (3.4) and $\lambda \in (0, +\infty)$.

Then boundary value problem (1.1) has at least one positive solution y with $0 < \|y\| < r$.

Proof In view of (H_4) , we choose $n_0 \in \{1, 2, \dots\}$ such that

$$\begin{aligned}r &> \frac{\lambda}{\Gamma(\alpha)} \left(\int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \right. \\ &\quad \left. + \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{k}{2}(s-\tau)^2\right) + h_1(r) \right) ds \right) + \frac{1}{n_0}.\end{aligned}$$

Let $N_0 = \{n_0, n_0 + 1, \dots\}$. Fix $n \in N_0$ and consider the family of integral equations

$$y(t) = \begin{cases} \kappa \int_0^1 \lambda G^*(t, s) (f_n(s, y^*(s-\tau)) + \rho(s)) ds + \frac{1}{n}, & t \in (0, 1), \\ \frac{1}{n}, & t \in [-\tau, 0], \end{cases} \quad (3.16)$$

where $\kappa \in (0, 1)$,

$$f_n(t, y^*(t-\tau)) + \rho(t) = \begin{cases} f(t, y^*(t-\tau)) + \rho(t), & y^*(t-\tau) \geq \frac{1}{n}, \\ f(t, \frac{1}{n}) + \rho(t), & y^*(t-\tau) < \frac{1}{n}. \end{cases}$$

We claim that any solution y of (3.16) for any $\kappa \in (0, 1)$ must satisfy $\|y\| \neq r$. Otherwise, assume that y is a solution of (3.16) for some $\kappa \in (0, 1)$ such that $\|y\| = r$. Then $y^*(t - \tau) \geq \frac{1}{n}$ for $t \in (0, 1)$. In view of Lemma 2.3, we have

$$\|y\| \leq \frac{\kappa \lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} (f_n(s, y^*(s-\tau)) + \rho(s)) ds + \frac{1}{n}. \quad (3.17)$$

Thus, for $t \in (0, 1)$, we have

$$\begin{aligned} y(t) &\geq \frac{1}{n} + \frac{\kappa \lambda t^4}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} (f_n(s, y^*(s-\tau)) + \rho(s)) ds \\ &\geq \frac{1}{n} + t^{\alpha-1} \left(\|y\| - \frac{1}{n} \right) \\ &\geq (1 - t^{\alpha-1}) \frac{1}{n} + t^{\alpha-1} \|y\| \\ &\geq t^{\alpha-1} \|y\| \geq t^2 r. \end{aligned}$$

Then like for (3.6), for $t \in (0, 1)$, we have

$$\begin{aligned} y(t) - t^{5-\alpha} \omega(t) &\geq t^2(r - c) \\ &\geq \frac{r}{2} t^2. \end{aligned}$$

Then from (H_1) , for $t \in (0, 1)$, $\kappa \in (0, 1)$, we have

$$\begin{aligned} y(t) &= \frac{1}{n} + \kappa \lambda \int_0^1 G^*(t, s) (f_n(s, y^*(s-\tau)) + \rho(s)) ds \\ &= \frac{1}{n} + \kappa \lambda \int_0^1 G^*(t, s) (f(s, y^*(s-\tau)) + \rho(s)) ds \\ &\leq \frac{1}{n} + \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} (f(s, y^*(s-\tau)) + \rho(s)) ds \\ &\leq \frac{1}{n_0} + \frac{\lambda}{\Gamma(\alpha)} \int_0^\tau s(1-s)^{\alpha-2} (\varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau)))) ds \\ &\quad + \frac{\lambda}{\Gamma(\alpha)} \int_\tau^1 s(1-s)^{\alpha-2} \\ &\quad \times \left(\varphi_1(s) \left(g\left(\frac{r}{2}(s-\tau)^2\right) + h_1(y(s-\tau) - (s-\tau)^{5-\alpha} \omega(s-\tau)) \right) \right) ds. \end{aligned}$$

Hence we obtain

$$\begin{aligned} r &= \|y(t)\| \\ &\leq \frac{1}{n_0} + \frac{\lambda}{\Gamma(\alpha)} \int_0^\tau s(1-s)^{\alpha-2} (\varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau)))) ds \\ &\quad + \frac{\lambda}{\Gamma(\alpha)} \int_\tau^1 s(1-s)^{\alpha-2} \\ &\quad \times \left(\varphi_1(s) \left(g\left(\frac{r}{2}(s-\tau)^2\right) + h_1(y(s-\tau) - (s-\tau)^{5-\alpha} \omega(s-\tau)) \right) \right) ds. \end{aligned}$$

This is a contradiction and the claim is proved.

Now the Leray-Schauder nonlinear alternative theorem guarantees that the equation

$$y(t) = \int_0^1 \lambda G^*(t, s) (f_n(s, y^*(s - \tau)) + \rho(s)) ds + \frac{1}{n}$$

has a solution y_n , in $\bar{\Omega}_r = \{y \in C[0, 1] : \|y\| \leq r\}$, for $t \in (0, 1)$.

Next we claim that $y_n(t)$ has a uniform sharper lower bound. In view of (H_1) and $\|y_n(t)\| \leq r$, we obtain

$$\begin{aligned} y_n(t) &= \frac{1}{n} + \lambda \int_0^1 G^*(t, s) (f_n(s, y_n^*(s - \tau)) + \rho(s)) ds \\ &\geq \frac{1}{n} + \lambda \int_\beta^\gamma G^*(t, s) (f(s, y_n^*(s - \tau)) + \rho(s)) ds \\ &\geq \frac{\lambda t^4}{\Gamma(\alpha)} \int_\beta^\gamma s(1-s)^{\alpha-2} (f(s, y_n^*(s - \tau)) + \rho(s)) ds \\ &\geq \frac{\lambda t^4}{\Gamma(\alpha)} \int_\beta^\gamma s(1-s)^{\alpha-2} \left(\varphi_2(s) h_2\left(\frac{r}{2}(s - \tau)^2\right) \right) ds \\ &\geq \frac{\lambda t^4}{\Gamma(\alpha)} h_2\left(\frac{r \zeta_1}{2}\right) \int_\beta^\gamma s(1-s)^{\alpha-2} \varphi_2(s) ds. \end{aligned}$$

Choosing $\delta(t) = \frac{\lambda t^4}{\Gamma(\alpha)} h_2\left(\frac{r \zeta_1}{2}\right) \int_\beta^\gamma s(1-s)^{\alpha-2} \varphi_2(s) ds$. Then we conclude that there exists a function $\delta \in C(0, 1)$ that is unrelated to n such that $\delta(t) > 0$ for a.e. $t \in (0, 1)$ and for any $n \in N_0$,

$$y_n(t) \geq \delta(t).$$

Then we prove $\{y_n\}_{n \in N_0}$ is an equicontinuous family on $(0, 1)$. Since G^* is uniformly continuous for $t \in (0, 1)$, that is, for any $\epsilon > 0$, there exists $\zeta_0 > 0$, when $t_1, t_2 \in [0, 1]$ and $|t_1 - t_2| < \zeta_0$, we have

$$\begin{aligned} |G^*(t_1, s) - G^*(t_2, s)| &= \frac{\epsilon}{2\lambda} \left(\int_0^\tau \varphi_1(s) (g(\eta(s - \tau)) + h_1(\eta(s - \tau))) ds \right. \\ &\quad \left. + \int_\tau^1 \varphi_1(s) \left(g\left(\frac{k}{2}(s - \tau)^2\right) + h_1(r) \right) ds \right)^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} |(y_n)(t_1) - (y_n)(t_2)| &\leq \lambda \int_0^\tau |G^*(t_1, s) - G^*(t_2, s)| \varphi_1(s) (g(\eta(s - \tau)) + h_1(\eta(s - \tau))) ds \\ &\quad + \lambda \int_\tau^1 |G^*(t_1, s) - G^*(t_2, s)| \varphi_1(s) \left(g\left(\frac{k}{2}(s - \tau)^2\right) + h_1(r) \right) ds \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $\{y_n\}_{n \in N_0}$ is an equicontinuous family on $(0, 1)$. By the Arzelà-Ascoli theorem, there exist a subsequence N_1 of N_0 and $y \in C(0, 1)$ such that $\{y_n\}_{n \in N_1}$ is uniformly convergent to y and y satisfies $\delta(t) \leq y(t) \leq r$ for any $t \in (0, 1)$. By the Lebesgue dominated

convergence theorem, in view of

$$y_n(t) = \int_0^1 \lambda G^*(t, s) (f_n(s, y_n^*(s - \tau)) + \rho(s)) ds,$$

we have

$$y(t) = \int_0^1 \lambda G^*(t, s) (f(s, y^*(s - \tau)) + \rho(s)) ds.$$

Then boundary value problem (1.1) has one positive solution with $0 < \|y\| < r$. The proof is completed. \square

Theorem 3.4 *Let (H_1) and (H_2) hold. Assume that there exists a subinterval $[\beta, \gamma] \subset (\tau, 1)$ satisfying*

$$\begin{aligned} \max g(\cdot) + \max h_1(\cdot) &\leq \frac{r_2}{\frac{\lambda}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} \varphi_1(s) ds}, \\ \min_{0 \leq y \leq r_1} h_2(y) &\geq \frac{r_1}{\frac{\lambda \zeta_2}{\Gamma(\alpha)} \int_\beta^\gamma s(1-s)^{\alpha-2} \varphi_2(s) ds}, \end{aligned}$$

where $\lambda \in (0, +\infty)$. Then boundary value problem (1.1) has at least one positive solution.

Proof In view of Theorem 3.1, for $y \in \partial\Omega_2$, $t \in [0, 1]$, we obtain

$$y(t) - t^{5-\alpha} \omega(t) \geq \frac{r_2}{2} t^2. \quad (3.18)$$

Then from (H_1) , (3.18), and Lemma 2.3, we get

$$\begin{aligned} (Ty)(t) &\leq \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \\ &\quad + \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) \\ &\quad \times \left(g\left(\frac{r_2}{2}(s-\tau)^2\right) + h_1(y(s-\tau) - (s-\tau)^{5-\alpha} \omega(s-\tau)) \right) ds \\ &\leq \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \\ &\quad + \frac{\lambda t^{5-\alpha}}{\Gamma(\alpha)} \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{r_2}{2}(s-\tau)^2\right) + h_1(y(s-\tau)) \right) ds \\ &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) (g(\eta(s-\tau)) + h_1(\eta(s-\tau))) ds \\ &\quad + \frac{\lambda}{\Gamma(\alpha)} \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) \left(g\left(\frac{k}{2}(s-\tau)^2\right) + h_1(r_2) \right) ds \\ &\leq \frac{\lambda}{\Gamma(\alpha)} \frac{r_2}{\frac{\lambda}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} \varphi_1(s) ds} \int_0^\tau s(1-s)^{\alpha-2} \varphi_1(s) ds \\ &\quad + \frac{\lambda}{\Gamma(\alpha)} \frac{r_2}{\frac{\lambda}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} \varphi_1(s) ds} \int_\tau^1 s(1-s)^{\alpha-2} \varphi_1(s) ds \leq r_2. \end{aligned}$$

Therefore, for $y \in \partial\Omega_2$, we have $\|Ty\| \leq \|y\|$.

On the other hand, for $y \in \partial\Omega_1$, like for (3.6), for $t \in [0, 1]$, we obtain

$$\begin{aligned} y(t) - t^{5-\alpha}\omega(t) &\geq t^2(r_1 - c) \\ &\geq \frac{r_1}{2}t^2. \end{aligned} \quad (3.19)$$

Thus from (H_1) , (3.19), and Lemma 2.3, we get

$$\begin{aligned} \|Ty\| &\geq \int_{\beta}^{\gamma} \lambda \min_{t \in [\beta, \gamma]} G^*(t, s) (f(s, (s-\tau)^{\alpha-5}y(s-\tau) - \omega(s-\tau)) + \rho(s)) ds \\ &\geq \int_{\beta}^{\gamma} \lambda \min_{t \in [\beta, \gamma]} G^*(t, s) \varphi_2(s) h_2\left(\frac{r_1}{2}(s-\tau)^2\right) ds \\ &\geq \frac{\lambda \zeta_2}{\Gamma(\alpha)} \frac{r_1}{\frac{\lambda \zeta_2}{\Gamma(\alpha)} \int_{\beta}^{\gamma} s(1-s)^{\alpha-2} \varphi_2(s) ds} \int_{\beta}^{\gamma} s(1-s)^{\alpha-2} \varphi_2(s) ds \\ &\geq r_1. \end{aligned}$$

Therefore, for $y \in \partial\Omega_1$, we have $\|Ty\| \geq \|y\|$. Arguments similar to those at the end of the proof of Theorem 3.1 show that boundary value problem (1.1) has a positive solution. The proof is completed. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

¹Department of Basic Courses, Shandong Women's University, Jinan, Shandong 250300, P.R. China. ²School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, P.R. China.

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